

THE LEONTOVICH BOUNDARY CONDITIONS AND CALCULATION OF EFFECTIVE IMPEDANCE OF INHOMOGENEOUS METAL

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Abstract

We bring forward rather simple algorithm allowing us to calculate the effective impedance of inhomogeneous metals in the frequency region where the local Leontovich (the impedance) boundary conditions are justified. The inhomogeneity is due to the properties of the metal or/and the surface roughness. Our results are nonperturbative ones with respect to the inhomogeneity amplitude. They are based on the recently obtained exact result for the effective impedance of inhomogeneous metals with flat surfaces [1, 2]. One-dimension surfaces inhomogeneities are examined. Particular attention is paid to the influence of generated evanescent waves on the reflection characteristics. We show that if the surface roughness is rather strong, the element of the effective impedance tensor relating to the p- polarization state is much greater than the input local impedance. As examples, we calculate: i) the effective impedance for a flat surface with strongly nonhomogeneous periodic strip-like local impedance; ii) the effective impedance associated with one-dimensional lamellar grating. For the problem (i) we also present equations for the forth lines of the Pointing vector in the vicinity of the surface.

Keywords: Inhomogeneous metals, surface impedance

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1 INTRODUCTION

It is well known that to solve an electrodynamic problem external with respect to a metal, one can use the impedance boundary conditions [3]

$$\mathbf{E}_t = \hat{\zeta}[\mathbf{n}, \mathbf{H}_t], \quad (1)$$

\mathbf{E}_t and \mathbf{H}_t are the tangential components of electric and magnetic vectors at the metallic surface; \mathbf{n} is the unit external normal vector to the surface of the metal; $\hat{\zeta}$ is the two-dimensional surface impedance tensor. Its real and imaginary parts define the dissipated energy and the phase shift of the reflected electromagnetic wave respectively (see, for example, [3]).

If for an inhomogeneous metal the penetration depth of an electromagnetic wave δ is much less than a characteristic size a of the surface inhomogeneity, omitting the terms of the order of $|\zeta a| \delta / a$, the local values of the impedance can be used in the boundary conditions (1). In this case Eqs.(1) are usually called the local Leontovich (the impedance) boundary conditions [4]. The surface inhomogeneity can be due to the inhomogeneity of the properties of the metal or/and the surface roughness. The elements of the local surface impedance tensor of an inhomogeneous metal are functions of position on the surface. If the metal surface is rough, the components of the unit vector \mathbf{n} depend on position too. The more complex is the geometry and composition of the surface, the more complex is the solution of the problem. When the surface inhomogeneity is strong, the solution requires cumbersome numerical calculations.

The structure of electromagnetic fields in the immediate vicinity of the inhomogeneous metallic surface is very complex too. The fields above the surface are represented by incident, reflected, scattered and evanescent waves. However, sometimes it is sufficient to know the reflected electromagnetic wave only, not going into details of the field distribution near the surface.

In what follows, when calculating we assume that

$$\delta \ll a \ll \lambda, \quad (2)$$

where $\lambda = 2\pi c/\omega$ is the vacuum wavelength. When inequalities (2) are fulfilled, only the fields averaged over the surface inhomogeneities "survive" far (at the distance much greater than the vacuum wavelength) from a stochastically or periodically inhomogeneous metallic surface. To calculate these fields, it is

convenient to introduce the effective surface impedance relating the tangential averaged fields at a plane close to the real metallic surface (or coinciding with this surface, if the surface of the metal is flat).

We define the effective impedance tensor with the aid of equations similar to Eqs.(1) for a homogeneous metal with the flat surface, let us say, $x_3 = 0$:

$$\langle \mathbf{E}_t \rangle = \hat{\zeta}_{ef}[\mathbf{e}_3, \langle \mathbf{H}_t \rangle]; \quad (3)$$

\mathbf{e}_3 is the unit normal vector to this plane, $\langle \cdot \rangle$ denote an average over the plane $x_3 = 0$. In the case of a stochastically inhomogeneous surface the averaging is carried out over the ensemble of realizations of the inhomogeneities; if the inhomogeneity is a periodic one, the brackets denote the average over the period. If the tensor $\hat{\zeta}_{ef}$ is known, when calculating the averaged electromagnetic fields we return to the reflection problem from a homogeneous flat surface.

The general way providing the correct result for the effective impedance is to derive equations for the averaged electromagnetic fields both in the metal and in the medium over the metal (we assume it to be vacuum), as well as the boundary conditions for these fields at the metal-vacuum interface. By proceeding from these equations the tensor $\hat{\zeta}_{ef}$ can be calculated.

The similar method is used rather often when calculating effective characteristics of inhomogeneous media. It goes back to the pioneering works of I.M.Lifshitz and his co-authors¹. However, mostly this regular method is applicable when the inhomogeneity amplitude is small and the perturbation theory can be used (with regard to the calculation of the effective impedance see [6, 7, 8]). The exact solutions for effective characteristics of strongly inhomogeneous media can be found very rarely. One of examples is the calculation of the effective static conductivity of some two- dimensional inhomogeneous metals [9].

The other example is a quite new result for the effective surface impedance of an inhomogeneous metal with flat surface [1, 2]. It is valid in the frequency region of the local impedance boundary conditions applicability. Recently this solution was used when calculating the effective impedance of strongly anisotropic polycrystalline metals both under the conditions of normal [1, 2] and anomalous skin effect [10].

In [1, 2] the expression for the effective impedance was obtained with the aid of the general scheme. Herein we present much more straightforward reasoning leading to the same result and extend it to the case of rough metallic surfaces. This result is a nonperturbative one with respect to the inhomogeneity amplitude. It is exact up to the limits of the local impedance boundary conditions (1) applicability.

The organization of the paper is as follows. In Section 2 some general remarks useful when calculating the effective impedance are given. In Section 3 the simple way to derive the expression for ζ_{ef} in the case of flat inhomogeneous surfaces is presented. As an example, we analyze in details the case of one-dimensional strongly nonhomogeneous periodic strip-like local impedance. In Section 4 we present the algorithm allowing us to calculate the effective impedance in the case of one-dimensional rough surface. The obtained formulas are used to calculate the effective impedance associated with a lamellar grating. The s- and p-polarization states are examined separately. Concluding remarks are given in Section 5.

2 PRELIMINARY REMARKS

Suppose the local impedance tensor $\hat{\zeta}$ is a known. A plane wave² $\mathbf{E}^i = \mathbf{E}_0 e^{-i(kx_3 + \omega t)}$ is incident upon the surface from the region $x_3 > 0$. The metal occupies the volume beneath the plane $x_3 = 0$. The surface of the metal is assumed to be the plane $x_3 = 0$ itself, or it is the plane $x_3 = 0$ perturbed by a roughness. For the sake of definiteness we suppose that the surfaces does not stand out over the plane $x_3 = 0$. Suppose the surface inhomogeneity is a stochastic or a periodic one. For a rough surface its profile is defined.

Let $\mathbf{E}(\mathbf{x})$ and $\mathbf{H}(\mathbf{x})$ be the total electromagnetic vectors in the region $x_3 > 0$. We extract the averaged electric and magnetic vectors:

$$\mathbf{E}(\mathbf{x}) = \langle \mathbf{E}(\mathbf{x}) \rangle + \mathbf{e}(\mathbf{x}); \quad \mathbf{H}(\mathbf{x}) = \langle \mathbf{H}(\mathbf{x}) \rangle + \mathbf{h}(\mathbf{x}); \quad \langle \mathbf{e}(\mathbf{x}) \rangle = \langle \mathbf{h}(\mathbf{x}) \rangle = \mathbf{0}. \quad (4.1);$$

If $a \ll \lambda$, the fields $\mathbf{e}(\mathbf{x})$ and $\mathbf{h}(\mathbf{x})$ represent mainly evanescent waves damping at the distances of the order of a above the surface. When $x_3 \geq d$, where d is a distance complying with the inequality $a \ll d \ll \lambda$, the leading terms in the expressions for the total fields are equal to their averaged values:

$$\mathbf{E}(\mathbf{x}) \approx \langle \mathbf{E}(x_3) \rangle = \mathbf{E}_0 e^{-ikx_3} + \langle \mathbf{E}^r \rangle e^{ikx_3}; \quad \mathbf{H} \approx \langle \mathbf{H}(x_3) \rangle = \mathbf{H}_0 e^{-ikx_3} + \langle \mathbf{H}^r \rangle e^{ikx_3}; \quad (4.2)$$

¹The first one was [5]

²To calculate the surface impedance of a metal it is sufficient to investigate the normal incidence of an electromagnetic wave onto the metal surface (see, for example, [11])

$\langle \mathbf{E}^r \rangle$ and $\langle \mathbf{H}^r \rangle$ are the electric and magnetic vectors of the reflected wave respectively.

Suppose we are interested in the averaged fields only. Since no inhomogeneous parameters enter the Maxwell equations in the vacuum, the averaging of these equations provides the standard Maxwell equations for the averaged fields. Our problem is to obtain the boundary conditions for the averaged fields with the aid of the exact Eqs.(1). Comparing the last with Eqs.(3), we define the effective impedance. In what follows we solve this problem, reducing it to calculation of the magnetic field in the vicinity of infinitely conducting surface of the same geometry.

To start the calculation, we remind that the elements of the tensor $\hat{\zeta}$ are of the order of the ratio $\delta/\lambda \ll 1$. Consequently, in accordance with Eqs.(1), the tangential components of electric vector at the metallic surface are much less than the tangential components of magnetic vector.

Next, at the surface of a metal the magnetic vector \mathbf{H}_t is approximately equal to the magnetic vector \mathbf{H}_t^{per} at the surface of a perfect conductor ($|\zeta_{\alpha\beta}| = 0$) of the same geometry. The same is true in the immediate vicinity of the metallic surface. Then at the plane $x_3 = 0$

$$\langle \mathbf{H}_t \rangle \approx \langle \mathbf{H}_t^{per}(\mathbf{x}_{||}, 0) \rangle, \quad (5)$$

$\mathbf{x}_{||}$ is two-dimensional position vector at the plane $x_3 = 0$. With regard to Eqs.(1) the first nonvanishing term in the expression for the tangential electric vector \mathbf{E}_t at the metallic surface is

$$\mathbf{E}_t \approx \hat{\zeta}(\mathbf{x})[\mathbf{n}(\mathbf{x}), \mathbf{H}_t^{per}]. \quad (6.1)$$

If the surface is flat (it is the plane $x_3 = 0$ itself; $\mathbf{n} = \mathbf{e}_3$), the vector $\langle \mathbf{E}_t \rangle$ is obtained from Eq.(6.1) directly:

$$\langle \mathbf{E}_t \rangle = \langle \hat{\zeta}(\mathbf{x}_{||})[\mathbf{e}_3, \mathbf{H}_t^{per}] \rangle; \quad (6.2)$$

In [2] we showed that the corrections to the expressions (6.2) are of the order of $\delta^2/a\lambda\mathbf{E}_0$. These small terms cannot be taken into account in the framework of the local Leontovich boundary conditions and, consequently, the averaged electric field cannot be calculated more accurately.

In the case of a flat inhomogeneous metallic surface Eqs.(3), (5) and (6.2) allow us to calculate the effective impedance. The obtained result cannot be improved. It is exact up to the limits of local impedance boundary conditions applicability.

The equations (5) and (6.1) allow us to define the effective impedance for a rough metallic surface too. However, the method of calculation is not so straightforward (see Section 4).

3 FLAT SURFACE OF INHOMOGENEOUS METAL

When the aforesaid argumentation is taken into account, the result of [1, 2] for the effective impedance of an inhomogeneous metal with a flat surface,

$$\hat{\zeta}_{ef} = \langle \hat{\zeta}(\mathbf{x}_{||}) \rangle, \quad (7)$$

is almost obvious. Indeed, in the case of normal incidence of an electromagnetic wave upon the flat surface $x_3 = 0$ of a perfect conductor the magnetic vector at the surface is $\mathbf{H}_t^{per} = 2\mathbf{H}_0$, where \mathbf{H}_0 is the amplitude of the magnetic field of the incident wave. This vector does not depend on position at the surface. Thus, with the aid of Eqs.(5) and (6.2) we have:

$$\langle \mathbf{H}_t \rangle = 2\mathbf{H}_0; \quad \langle \mathbf{E}_t \rangle = 2 \langle \hat{\zeta} \rangle [\mathbf{e}_3, \mathbf{H}_0]. \quad (8)$$

Comparing Eqs.(8) with Eqs.(3), we obtain Eq.(7). This result is correct for any metal with inhomogeneous flat surface both under the conditions of normal and anomalous skin effect.

3.1 ONE-DIMENSIONAL STRIP-LIKE LOCAL IMPEDANCE

Suppose the local impedance tensor of an isotropic inhomogeneous metal is one-dimensional periodic strip-like function: $\hat{\zeta}_{ik}(\mathbf{x}_{||}) = \zeta(x_1)\delta_{ik}$ ($i, k = 1, 2$), where

$$\zeta(x_1) = \begin{cases} \zeta_1; & |x_1| < a, \\ \zeta_2; & a < |x_1| < b, \end{cases} \quad (9)$$

$2b$ is the period. All the parameters, namely, a , b , $b - a$ and the local impedances ζ_1 , ζ_2 are compatible with inequalities (2).

We use this example not only to illustrate how simple the effective impedance can be calculated with the aid of Eq.(7), but also to visualize the difference between the fields in the immediate vicinity of one-dimensional inhomogeneous metallic surface for the s- and p- polarization states.

With regard to Eq.(7) for both polarizations the effective impedance is

$$\zeta_{ef} = \zeta_1 \frac{a}{b} + \zeta_2 \left(1 - \frac{a}{b}\right). \quad (10)$$

However, the difference between the polarizations exhibits itself in the values of the fields near the surface. To complete the picture, for both polarizations we calculate the fields above the metal and the time- averaged Poynting vector $\bar{\mathbf{S}} = c[\mathbf{E}, \mathbf{H}]/4\pi$ defining the time-averaged energy flux. The continuity equation for this vector, $\text{div}\bar{\mathbf{S}} = 0$, represents the energy conservation law (see, for example, [3]).

P-POLARIZATION

Let the magnetic vector of the incident wave be $\mathbf{H}^i(0, H_0, 0) \exp[-ikx_3]$. The only nonzero component of the total magnetic vector above the metal can be written as

$$H_2(x_1, x_3) = H_0 \left\{ e^{-ikx_3} + h_0 e^{ikx_3} + h(x_1, x_3) \right\}, \quad (11.1)$$

where h_0 is the amplitude of the reflected wave and

$$h(x_1, x_3) = \sum_{-\infty; q \neq 0}^{\infty} h_q e^{(i\pi q x_1/b - \alpha_q x_3)}; \quad \alpha_q = \sqrt{(q\pi/b)^2 - k^2} \approx \frac{q\pi}{b}. \quad (11.2)$$

represents the evanescent waves. With regard to the Maxwell equations the components of the electric vector are

$$E_1(x_1, x_3) = H_0 \left\{ \frac{1}{k} \sum_{-\infty; q \neq 0}^{\infty} h_q \alpha_q e^{(i\pi q x_1/b - \alpha_q x_3)} + h_0 e^{ikx_3} - e^{(-ikx_3)} \right\}; \quad (12.1)$$

$$E_3(x_1, x_3) = -H_0 \frac{\pi}{kb} \sum_{-\infty; q \neq 0}^{\infty} q h_q e^{(i\pi q x_1/b - \alpha_q x_3)}. \quad (12.2)$$

We see that to calculate the components of the electric vector $\mathbf{E}(x_1, x_3)$ up to the terms of the order of ζ allowed in the framework of the local impedance boundary conditions applicability, we need to know the amplitudes of the evanescent waves h_q ($q \neq 0$) up to the terms of the order $(kb)\zeta \ll \zeta$.

According to Eqs.(1) in the p-polarization state $E_1(x_1, 0) = -\zeta(x_1)H_2(x_1, 0)$. We Fourier analyze this equation making use of Eqs.(11) and (12.1). Then, omitting the terms of the order of ζ^2 , we obtain the coefficients h_q :

$$h_0 = 1 - 2\zeta_{ef}; \quad h_q|_{q \neq 0} = \frac{2i}{\pi|q|} (ka)(\zeta_1 - \zeta_2)F(\pi qa/b), \quad (13)$$

where $F(x) = \sin x/x$. As a result up to the terms of the order of ζ

$$H_2(x, y) = 2H_0 \left\{ \cos(kby/\pi) - \zeta_{ef} e^{(ikby/\pi)} \right\}; \quad (14.1)$$

$$E_1(x, y) = 2H_0 \left\{ i \sin(kby/\pi) - \zeta_{ef} e^{i(kby/\pi)} - \frac{2}{\pi} (\zeta_1 - \zeta_2) C_1(x, y) \right\}; \quad (14.2)$$

$$E_3(x, y) = \frac{4H_0}{\pi} (\zeta_1 - \zeta_2) S_1(x, y). \quad (14.3)$$

We introduce $x = \pi x_1/b$ and $y = \pi x_3/b$. The functions $C_m(x, y)$ and $S_m(x, y)$ are respectively the real and the imaginary parts of the function³ R_m of the complex argument $z = x + iy$:

$$R_m(z) = \sum_{q=1}^{\infty} \frac{\sin \pi qa/b}{q^m} e^{iqz}. \quad (15)$$

³It is easy to calculate the explicit form of the functions $R_m(z)$ for an arbitrary m , but trying to be short we do not present here the relevant expressions.

Note, all the corrections taking into account the finite conductivity of the metal are of the order of the input impedances ζ_1 and ζ_2 . Next, the expression for the magnetic field, Eq.(14.1), is just the same as if calculated for the p-polarized wave incident upon a homogeneous metallic surface with the impedance ζ_{ef} . However, the evanescent waves contribute to the electric field. In equations (14.2) and (14.3) they are represented by the terms including the functions $C_1(x, y)$ and $S_1(x, y)$. These terms are exponentially small when $y \gg 1$ (or $x_3 \gg b$).

From Eqs.(14) it follows that for the p-polarization state up to the terms of the order of ζ the Pointing vector $\overline{\mathbf{S}}^{(p)}$ (the superscript (p) indicates the polarization) can be written as

$$\overline{\mathbf{S}}^{(p)}(x, y) = -\frac{c|H_0|^2}{\pi} \nabla F_p(x, y), \quad (16)$$

where

$$F_p(x, y) = \frac{y}{2} \text{Re}\zeta_{ef} - \frac{\text{Re}(\zeta_1 - \zeta_2)}{\pi} \cos(kby/\pi) C_2(x, y). \quad (17)$$

Far from the surface ($y \gg 1$) the component $\overline{S}_1^{(p)}$ of the Pointing vector is exponentially small, and the component $\overline{S}_3^{(p)} = -(c/2\pi)|H_0|^2 \text{Re}\zeta_{ef}$. At the surface $\overline{S}_1^{(p)} = 0$ and the component $\overline{S}_3^{(p)}$ is defined by the local impedance: $\overline{S}_3^{(p)} = -(c/2\pi)|H_0|^2 \text{Re}\zeta_1$, if $|x_1| < a$ and $\overline{S}_3^{(p)} = -(c/2\pi)|H_0|^2 \text{Re}\zeta_2$, if $a < |x_1| < b$.

It is easy to see that along with Eq.(16) the components of the vector $\overline{\mathbf{S}}^{(p)}$ can be written as

$$\overline{\mathbf{S}}^{(p)} = \frac{c|H_0|^2}{\pi} \text{rot} \mathbf{A}_p(x, y), \quad (18.1)$$

where the vector $\mathbf{A}_p = -(0, A_p, 0)$ and

$$A_p(x, y) = \frac{x}{2} \text{Re}\zeta_{ef} + \frac{\text{Re}(\zeta_1 - \zeta_2)}{\pi} \cos(kby/\pi) S_2(x, y). \quad (18.2)$$

From Eq.(16) it follows that in the p-polarization state in addition to equation $\text{div} \overline{\mathbf{S}}^{(p)} = 0$ we also have $\text{rot} \overline{\mathbf{S}}^{(p)} = 0$. Thus, in the p-polarization state the vector $\overline{\mathbf{S}}^{(p)}$ satisfies the same set of equations as the electric vector in two-dimensional electrostatic problems. Making use of this analogy we introduce a complex potential $w(z) = F_p(x, y) - iA_p(x, y)$ for the Pointing vector $\overline{\mathbf{S}}^{(p)}$:

$$w(z) = -\frac{c|H_0|^2}{\pi} \left\{ \frac{iz}{2} \text{Re}\zeta_{ef} + \text{Re}(\zeta_1 - \zeta_2) R_2(z) \right\}. \quad (19)$$

At the plane (x, y) equations $\text{Re} w(z) = \text{const}$ define the equipotential lines for the vector $\overline{\mathbf{S}}^{(p)}$, and equations $\text{Im} w(z) = \text{const}$ define the force lines of this vector. Evidently, far from the surface ($y \gg 1$) the force lines are parallel to the y axis. Near the surface ($y \leq 1$) they are distorted due to the influence of evanescent waves.

As an illustration, let us examine the limiting case of very strong inhomogeneity: $a/b \ll 1$, $|\zeta_1|/|\zeta_2| \gg 1$ and $(a/b)|\zeta_1| \gg |\zeta_2|$. With regard to Eq.(10) the relevant effective impedance is $\zeta_{ef} = (a/b)\zeta_1$, and according to Eq.(19) the complex potential is

$$w(z) = \frac{c|H_0|^2}{\pi} \text{Re}\zeta_{ef} \left\{ -\frac{i}{2} z + \ln(1 - e^{iz}) \right\}. \quad (20.1)$$

The force lines corresponding to this potential are given by equations

$$\tan(x/2) = \tan(x_\infty/2) \frac{1 - e^{-y}}{1 + e^{-y}}, \quad (20.2)$$

where x_∞ defines the position of the given line far from the surface. These force lines are shown in Fig.1 (full lines). All the lines being distributed uniformly far from the surface, meet at the point $x = 0$ on the surface. Immediately near the surface the force lines are the straight lines $y = \alpha x$ with $\alpha = \cot(x_\infty/2)$.

It can be easily verified that the complex potential defined by Eq.(20.1) is the same as the electrostatic complex potential of the system of charged straight lines that are perpendicular to the plane (x_1, x_3) and intersecting this plane at the points $x_1 = 2bn$; $n = 0, \pm 1, \pm 2 : \dots$

S-POLARIZATION

Let the electric vector of the incident wave be $\mathbf{E}^i = (0, E_0, 0) \exp[-ikx_3]$. The total electric vector above the metal is

$$E_2(x_1, x_3) = E_0 \{e^{-ikx_3} + e_0 e^{ikx_3} + e(x_1, x_3)\}, \quad e(x_1, x_3) = \sum_{-\infty; q \neq 0}^{\infty} e_q e^{(i\pi q x_1/b - \alpha_q x_3)}, \quad (21)$$

α_q is defined in Eq.(11.2); e_0 corresponds to the reflected wave and $e(x_1, x_2)$ represents the evanescent waves.

To define the Fourier coefficients e_0 and e_q we use the boundary conditions (1). Omitting the terms of the order of ζ^2/kb , we obtain

$$e_0 = -(1 - 2\zeta_{ef}); \quad e_q|_{q \neq 0} = \frac{2a}{b}(\zeta_1 - \zeta_2)F(\pi qa/b). \quad (22)$$

(compare with Eq.(13)). Then in terms of the variables x and y our result for the fields above the surface is

$$E_2(x, y) = 2E_0 \left\{ -i \sin(kby/\pi) + \zeta_{ef} e^{ikby/\pi} + \frac{2}{\pi}(\zeta_1 - \zeta_2)C_1(x, y) \right\}, \quad (23.1)$$

$$H_1(x, y) = 2E_0 \left\{ \cos(kby/\pi) - \zeta_{ef} e^{ikby/\pi} - \frac{2i(\zeta_1 - \zeta_2)}{kb} C_0(x, y) \right\}, \quad (23.2)$$

$$H_3(x, y) = 4iE_0 \frac{(\zeta_1 - \zeta_2)}{kb} S_0(x, y); \quad (23.3)$$

In contrast to the p-polarization state the evanescent waves contribute both to the electric and magnetic fields near the metallic surface. Next, the contributions of evanescent waves to the components of the magnetic vector are of the order of $\zeta/kb \gg \zeta$ (compare with Eqs.(14)).

Of course, in accordance with the energy conservation law in the s-polarization state the components of the time-averaged Poynting vector satisfy the equation $\text{div} \overline{\mathbf{S}^{(s)}} = 0$. However, the curl of this vector is not equal to zero and, consequently, in the s-polarization state no complex potential can be introduced.

To define the equation for the force lines of the vector $\overline{\mathbf{S}^{(s)}}$ we write it as

$$\overline{\mathbf{S}^{(s)}} = \frac{c|E_0|^2}{\pi} \text{rot} \mathbf{A}_s(x, y), \quad \mathbf{A}_s = -(0, A_s, 0); \quad (24)$$

$$A_s(x, y) = \frac{x}{2} \text{Re} \zeta_{ef} + \frac{\text{Re}(\zeta_1 - \zeta_2)}{kb} \left[\sin(kby/\pi) S_1(x, y) + \frac{kb}{\pi} \cos(kby/\pi) S_2(x, y) \right] \quad (25)$$

(compare with Eqs.(18)). Then the force lines of the vector $\overline{\mathbf{S}^{(s)}}$ are given by equations $A_s = \text{const}$. In the case of very strong inhomogeneity ($a/b \ll 1$, $|\zeta_1| \gg |\zeta_2|$ and $(a/b)|\zeta_1| \gg |\zeta_2|$), they are described by the equation

$$\arctan u - 2 \frac{u}{(1+u^2)} \frac{ye^{-y}}{(1-e^{-2y})} = \frac{\pi - x_\infty}{2}, \quad u(x, y) = \frac{1 - e^{-y}}{1 + e^{-y}} \cot(x/2) \quad (26.1)$$

(compare with Eq.(20.2)). Again x_∞ defines the position of the line far from the surface. Immediately near the surface the force lines are the straight lines $y = u_0 x$. For the given value x_∞ the slope u_0 is defined by the equation

$$\arctan u_0 - \frac{u_0}{1+u_0^2} = \frac{\pi - x_\infty}{2}. \quad (26.2)$$

These force lines are shown in Fig.1 (dashed lines).

Summarizing the results of this section we would like to emphasize that the main difference between the two polarizations is the presence of the terms of the order of $\zeta/kb \sim \delta/b$ in the expressions for the components of the magnetic vector in the s-polarization state.

4 1-D ROUGH SURFACE

In this Section we show that if the metallic surface is rough, to calculate the effective impedance it is sufficient to know the magnetic vector \mathbf{H}^{per} near the surface of a perfect conductor of the same geometry. The main problem is to calculate the components of this vector. In the general case they depend on the surface geometry and, as a rule, on position at the surface.

We restrict ourselves with one-dimensional rough homogeneous metallic surfaces only. To be definite, let us suppose that our one-dimensional surface is a periodic one, $2b$ is the period. A part of the surface is the plane $x_3 = 0$ itself, and each period has a deepening with a throat of the length $2a$ (Fig.2a).

In the case of the p-polarization state we calculate the relevant element of the effective impedance tensor for an arbitrary surface profile. For the s-polarization state we perform the calculation for one-dimensional lamellar gratings. We use this example to show the difference between the polarizations.

To show the method of calculation, let us suppose that we know the vector \mathbf{H}^{per} for a given surface profile. With regard to the definition (3) of ζ_{ef} and Eq.(4.2) we have that far from the surface ($x_3 \gg \lambda$) the fields are the same as the ones above a flat metallic surface with the impedance equal to ζ_{ef} . This means that far from the surface the only nonzero component of the time-averaged Poynting vector is

$$\overline{S}_3 = -\frac{c|H_0|^2}{2\pi} \text{Re}\zeta_{ef} \quad (x_3 \gg \lambda), \quad (27.1)$$

where H_0 is the amplitude of the magnetic field in the incident wave. On the other hand, since near a metallic surface the magnetic vector \mathbf{H} is nearly the same as \mathbf{H}^{per} , at the rough metallic surface itself the time-averaged Poynting vector is

$$\overline{\mathbf{S}} = -\frac{c\text{Re}\zeta}{8\pi} |\mathbf{H}_t^{per}(\mathbf{x})|^2 \mathbf{n}(\mathbf{x}), \quad (27.2)$$

\mathbf{n} is the unit external normal vector to the surface (see, for example, [3]). The energy conservation law requires $\text{div}\overline{\mathbf{S}} = 0$, and, consequently, the energy fluxes through the infinitely distant surface and the surface of the metal are equal. Then

$$\text{Re}\zeta_{ef} = \frac{\text{Re}\zeta}{8b|H_0|^2} \int |\mathbf{H}_t^{per}(\mathbf{x})|^2 dl, \quad (27.3)$$

the integration is carried out along one period of the line bounding the surface at the (x_1, x_3) plane.

The last equation enables us to calculate the real part of the effective impedance. To calculate $\text{Im}\zeta_{ef}$ we take into account that though no absorption happens when an electromagnetic wave reflects from a rough surface of the perfect conductor, a phase shift takes place. In other words, some pure imaginary surface impedance can be associated with the rough surface of the perfect conductor. It is easy to understand that its leading term is of the order of kb (we remind that $kb \ll 1$). Since $|\zeta| \sim k\delta \ll kb$, in the case of the finite conductivity the same pure imaginary surface impedance defines the leading term in the expression for $\text{Im}\zeta_{ef}$.

Now, supposing that $\zeta = 0$, with the aid of Eq.(3) we can define $\text{Im}\zeta_{ef}$ as the ratio of the tangential electric and magnetic fields averaged across the plane $x_3 = 0$. The result for the averaged tangential magnetic vector is rather obvious. Indeed, Eq.(3) have the form of the boundary conditions for a flat metallic surface. Consequently, the leading term in the expression for the averaged tangential magnetic field is $\langle \mathbf{H}_t \rangle = 2\mathbf{H}_0$. This statement is true both for a metallic surface and the surface of the perfect conductor. Next, when $\zeta = 0$, only the throat of the deepening contributes to the averaged electric field:

$$\langle \mathbf{E}_t^{per} \rangle = \frac{1}{2b} \int_{-a}^a dx_1 \mathbf{E}_t^{per}(x_1, x_3 = 0). \quad (28)$$

If the magnetic vector \mathbf{H}^{per} is known, the last integral can be calculated with the aid of the Stokes theorem or one of its modifications (see below).

Concluding this subsection we would like to note, that the components of the vector \mathbf{H}^{per} depend on the parameter $kb \ll 1$ (see Eq.(2)). We show that when calculating the leading terms of $\text{Re}\zeta_{ef}$ and $\text{Im}\zeta_{ef}$, we need to know the field \mathbf{H}^{per} up to the terms independent of this small parameter only.

In what follows p- and s- polarization states are examined separately.

4.1 CALCULATION OF $\zeta_{11}^{(ef)}$ (P-POLARIZATION STATE)

Let us start with the p-polarization state (the magnetic vector $\mathbf{H} = (0, H_2, 0)$ is parallel to the rulings). With regard to our notations the relevant element of the effective impedance tensor is ζ_{11}^{ef} .

It is easy to see that when a p-polarized wave is incident upon one-dimensional rough surface of a perfect conductor, and the characteristic sizes of the surface profile are compatible with inequalities (2), in the vicinity of the surface the expression for the magnetic field H_2^{per} can be written as $H_2^{per} = 2H_0 + (kb)h(x_1, x_3; kb)$; $kb \ll 1$. (In APPENDIX 1 we calculate H_2^{per} for an infinitely conducting lamellar grating.) As far as the terms of the order of $(kb)\zeta$ falls outside the framework of the local impedance boundary conditions applicability, the corrections $(kb)h(x_1, x_3; kb)$ has to be omitted when calculating $\text{Re}\zeta_{ef}$ with the aid of Eq.(27.3).

Next, to calculate the averaged tangential electric field $\langle \mathbf{E}_t^{per} \rangle = (\langle E_1^{(per)} \rangle, 0, 0)$ entering Eq.(28), we write the Stokes theorem:

$$\oint d\mathbf{S}[\nabla, \mathbf{E}^{per}] = \oint d\mathbf{r}\mathbf{E}^{per}. \quad (29)$$

The integral in the left-hand side of Eq.(29) is taken over the cross-section of the deepening ($d\mathbf{S} = (0, dS, 0)$); the integral in the right-hand side is carried out in the clockwise direction over the contour and the throat of the deepening. Now, taking into account that on the surface of the perfect conductor the tangential electric field is equal to zero, with the aid of the Maxwell equations we obtain

$$\langle E_1^{per} \rangle = \frac{ik}{2b} \int dS H_2^{per}(x_1, x_3). \quad (30)$$

The integral in the right-hand side of Eq.(30) is of the order of kSH_2/b , S is the area of the deepening. Since according to Eqs.(2) the value of $kS/b \ll 1$, when calculating the leading term in the expression for $\langle E_1^{(per)} \rangle$ we again use the approximation $H_2^{per}(x_1, x_3) = 2H_0$.

Let L_c be the length of the surface contour with respect to one period. Now, for an arbitrary shape of the deepening we can write the expression for the effective impedance leaving only the leading terms of $\text{Re}\zeta_{ef}$ and $\text{Im}\zeta_{ef}$:

$$\zeta_{11}^{(ef)} = \text{Re}\zeta \frac{L_c}{2b} + i \frac{kS}{2b}. \quad (31)$$

We see that independently of the shape and the size of the deepening $\text{Re}\zeta_{11}^{ef} > \text{Re}\zeta$ and $\text{Im}\zeta_{11}^{ef}$, being of the order of k multiplied by a characteristic size of the surface profile, is much greater than $\text{Im}\zeta \sim k\delta$. From Eq.(31) it follows that in the p-polarization state an intensification of absorption due to the surface roughness, is merely geometrical effect relating to an increase of the area of the absorbing surface.

Let us estimate the maximum value ζ_{max} of $\text{Re}\zeta_{11}^{ef}$. It is evident that ζ_{max} corresponds to the maximum permissible value of the ratio $L_c/2b$. Taking account of the inequalities (2), suppose the length L_c is of the order of the vacuum wavelength λ and the period $2b$ is of the order of the penetration depth δ . Then, since $|\zeta| \sim \delta/\lambda$, we have $\zeta_{max} \sim 1$. (We would like to recall that for good metals $|\zeta| \ll 1$.) Of course, this is only the upper bound of the permissible value of $\text{Re}\zeta_{11}^{ef}$. However, the estimate shows that when the length of the contour is much greater than the period ($L_c \gg 2b$), the value of $\text{Re}\zeta_{11}^{ef}$ exceeds $\text{Re}\zeta$ significantly. Such increase of the effective impedance has to manifest itself as giant absorption of the incident p-polarized wave.

The surface with $L_c \gg 2b$ can be realized, for example, in such a way. Suppose a planar metallic surface has periodic grooves. Suppose the boundary of the groove is a branching line (Fig.2.b). Let us say, this line has a fractal structure. Then the ratio $L_c/2b$ can be very large and, consequently, the value of the $\text{Re}\zeta_{11}^{(ef)}$ close to ζ_{max} can be achieved. When the throats of the grooves are sufficiently narrow, almost all absorption takes place inside "the pockets" beneath the plane $x_3 = 0$.

4.2 CALCULATION OF $\zeta_{22}^{(ef)}$ (S-POLARIZED WAVES)

It is much more difficult to calculate the effective impedance relating to the s-polarization state (the electric vector $\mathbf{E} = (0, E_2, 0)$ is parallel to the rulings). In our notations this is the element ζ_{22}^{ef} of the effective impedance tensor. The point is that for this polarization the magnetic vector \mathbf{H}^{per} varies significantly in the vicinity of the surface. Inside the grooves the strength of the magnetic field exponentially decays when the distance from the plane $x_3 = 0$ increases. Next, in contrast to the p-polarization

state, evanescent waves generated in the region $x_3 > 0$, contribute to the leading term of the tangential magnetic field H_1^{per} . As a result even for rather simple surfaces, such as lamellar grating, the value of $\zeta_{22}^{(ef)}$ can be calculated only numerically.

Let us write down the general expression for $\zeta_{22}^{(ef)}$. Suppose the surface of the metal is of the type shown in Fig.2a, and we know the vector \mathbf{H}_t^{per} in the vicinity of the surface. Then we can calculate $\text{Re}\zeta_{ef}$ with the aid of Eq.(27.3).

To calculate $\text{Im}\zeta_{ef}$ we need to know the averaged tangential magnetic $\langle H_1^{per}(x_1, 0) \rangle$ and electric $\langle E_2^{per}(x_1, 0) \rangle$ fields. Although the components of the magnetic vector depend on coordinates, with regard to the aforementioned arguments we can use the approximation $\langle H_1^{per}(x_1, 0) \rangle = 2H_0$ (H_0 is the amplitude of the magnetic vector of the incident wave). To calculate $\langle E_2^{per}(x_1, 0) \rangle$ we use the vector modification of the Stokes theorem [14] for the vector \mathbf{E}^{per} :

$$\oint [[d\mathbf{S}, \nabla], \mathbf{E}^{per}] = \oint [d\mathbf{r}, \mathbf{E}^{per}]. \quad (32.1)$$

The domains of integration are the same as when calculating the integral (29). Multiplying the vector equation (32.1) by the unit vector \mathbf{e}_3 with regard to the Maxwell equations and the boundary conditions for the vector \mathbf{E}^{per} we obtain

$$\langle E_2^{per}(x_1, 0) \rangle = -\frac{ik}{2b} \int dS H_1^{per}(x_1, x_3), \quad (32.2)$$

Now it is clear that in the s-polarization state the effective impedance is

$$\zeta_{22}^{(ef)} = \text{Re}\zeta \frac{L_c}{2b} Z - \frac{ikS}{2b} W, \quad (33.1)$$

where the leading terms of Z and W are

$$Z = \frac{1}{4L_c |H_0|^2} \int |\mathbf{H}_t^{per}(\mathbf{x})|^2 dl, \quad W = \frac{1}{2SH_0} \int dS H_1(x_1, x_3). \quad (33.2)$$

(compare with Eq.(31)). Again L_c is the length of the contour bounding our surface with respect to one period, and S is the area of the deepening. The factors Z and W show the difference between the polarizations.

4.3 $\zeta_{22}^{(ef)}$ ASSOCIATED WITH LAMELLAR GRATING

There are some regular methods allowing us to calculate the magnetic vector at the surface of a perfect conductor in the small roughness limit. However, if the surface roughness is not small, analytical solutions can be obtained for some specific surfaces only. Therefore, as an example, we examine an infinitely conducting lamellar grating (Fig.2c). We assume that all the sizes of the contour, namely the period $2b$, the width of the rectangular grooves $2a$, the depth of the grooves h as well as $b - a$, are in agreement with inequalities (2).

The electromagnetic fields in the vicinity of infinitely conducting lamellar gratings have been examined by a lot of authors (see, for example, [13]). Therefore we do not go into details of calculation, but present only the basic formulae used to obtain the result and some remarks relating to the computational procedure.

To calculate the vector H_t^{per} , following [13], we present the electric field in the central groove $E_2^{per} = E_2^{(-)}(x_1, x_3)$ ($|x_1| < a$; $-h < x_3 < 0$) as series of modal functions $\phi_n^{(s)}(x_1, x_3)$ that are the solutions of the Maxwell equations:

$$E_2^{(-)}(x_1, x_3) = \sum_{n=0}^{\infty} B_n^{(s)} \phi_n^{(s)}(x_1, x_3), \quad \phi_n^{(s)}(x_1, x_3) = \sin \frac{\pi n}{2a} (x_1 - a) \sin \beta_n (h + x_3), \quad (34.1)$$

$\beta_n^2 = k^2 - (\pi n/2a)^2$ and $\text{Re}\beta_n, \text{Im}\beta_n > 0$. The representation incorporates the boundary conditions for perfect conductors on the vertical and bottom horizontal facets of the groove.

In the region $x_3 > 0$ we seek the field $E_2^{per} = E_2^{(+)}(x_1, x_3)$ as

$$E_2^{(+)}(x_1, x_3) = \left\{ e^{-ikx_3} + \sum_{q=-\infty}^{\infty} e_q^+ e^{i(\pi qx_1/b + \alpha_q x_3)} \right\}; \quad (34.2)$$

$\alpha_0 = k$ and $\alpha_q = i\sqrt{(\pi q/b)^2 - k^2}$, if $q \neq 0$. The amplitude of the incident wave is equal to one, e_0 is the amplitude of the reflected wave and e_q ($q \neq 0$) are assigned to evanescent waves.

Taking account of the fields periodicity, matching the electric vector at the plane $x_3 = 0$ and the tangential magnetic vector at the throat of the central groove, after eliminating the amplitudes e_0 and e_q , we obtain a matrix equation for the coefficients $B_{2n-1}^{(s)}$. It has been repeatedly shown (see, for example, [13]) that in the case of the s-polarization state all the even coefficients B_{2n} are equal to zero.

Let us introduce a dimensionless parameter $\gamma = kb \ll 1$. From Eqs.(33) it follows that when calculating $\zeta_{22}^{(ef)}$ we need to know the components of the vector \mathbf{H}^{per} up to the terms independent of γ only. We seek the coefficients $B_{2n-1}^{(s)}$ ($n = 1, 2, \dots$) as series expansions in powers of γ . The first non-vanishing terms of these series are proportional to γ . To calculate the magnetic vector up to the terms independent of γ , we cut off the series at the first terms. In this approximation in place of $B_{2n-1}^{(s)}$ we introduce the coefficients Y_n :

$$B_{2n-1}^{(s)} = \frac{\gamma Y_n}{(2n-1) \sinh \chi_n}; \quad \chi_n = \frac{\pi(2n-1)h}{2a}. \quad (35)$$

The coefficients Y_n are the solution of the infinite set of equations:

$$\sum_{n=1}^{\infty} Y_n \Delta_{nm}(\mu) + \frac{\pi^2 \coth \chi_n}{4\mu(2m-1)} Y_m = \frac{2}{(2m-1)^2}; \quad (36.1)$$

$$\Delta_{nm}(\mu) = \sum_{q=1}^{\infty} \frac{(q\mu)[1 + \cos q\pi\mu]}{[(q\mu)^2 - (2n-1)^2][(q\mu)^2 - (2m-1)^2]}; \quad \mu = 2a/b. \quad (36.2)$$

Next, with $H_1^{(+)}(x_1, x_3)$ to denote $H_1^{per}(x_1, x_3)$ in the region above the plane $x_3 = 0$, and $\mathbf{H}^{(-)}(x_1, x_3)$ to denote $\mathbf{H}^{per}(x_1, x_3)$ in the central groove, in terms of the coefficients Y_n we have in the limit $\gamma \rightarrow 0$:

$$H_1^{(+)}(x_1, 0) = 2[1 + \delta H(x_1)], \quad (37.1)$$

$$\delta H(x_1) = \frac{2a}{b} \sum_{q=1}^{\infty} q \cos\left(\frac{\pi q x_1}{b}\right) \Phi\left(\frac{2a}{b}q\right), \quad \Phi(z) = \cos\left(\frac{\pi z}{2}\right) \sum_{n=1}^{\infty} \frac{Y_n}{[z^2 - (2n-1)^2]}. \quad (37.2)$$

Next,

$$H_1^{(-)}(x_1, x_3) = -\frac{\pi b}{2a} \sum_{n=1}^{\infty} \frac{Y_n}{\sinh \chi_n} \sin\left(\frac{\pi(2n-1)}{2a}(x_1 - a)\right) \cosh \chi_n \left(1 + \frac{x_3}{h}\right), \quad (37.3)$$

$$H_3^{(-)}(x_1, x_3) = \frac{\pi b}{2a} \sum_{n=1}^{\infty} \frac{Y_n}{\sinh \chi_n} \cos\left(\frac{\pi(2n-1)}{2a}(x_1 - a)\right) \sinh \chi_n \left(1 + \frac{x_3}{h}\right). \quad (37.4)$$

Note, from Eqs.(37.1) and (37.2) it immediately follows that $\langle \delta H(x_1) \rangle = 0$ and $\langle H_1^{(+)}(x_1, 0) \rangle = 2$.

Thus, at first we need to calculate the coefficient Y_n solving numerically the infinite set of equations (36). Then we use Eqs.(37) to calculate the components of the magnetic vector and define the real and imaginary parts of the effective impedance with the aid of Eqs.(33). We describe the computational algorithm in APPENDIX 2.

Let us write the coefficient Z entering Eq.(33.1) as

$$Z = Z_{pl} + Z_s, \quad (38.1)$$

where Z_{pl} and Z_s are related respectively to the parts of the plane $x_3 = 0$ incorporated in the contour and the inner facets of the grooves. With regard to Eqs.(37) we have

$$Z_{pl} = \frac{1}{(b+h)} \int_a^b |1 + \delta H(x_1)|^2 dx_1; \quad (38.2)$$

$$Z_s = \frac{1}{4(b+h)} \left\{ \int_{-h}^0 |H_3^{(-)}(a, x_3)|^2 dx_3 + \frac{1}{2} \int_0^a |H_1^{(-)}(x_1, -h)|^2 dx_1 \right\}. \quad (38.3)$$

Since the explicit forms of Z_{pl} and Z_s in terms of the coefficients Y_n are very lengthy, they are not presented here. The coefficient W defines the imaginary part of the effective impedance. For our lamellar grating

$$W = \frac{b}{\pi h} \sum_{n=1}^{\infty} \frac{Y_n}{(2n-1)^2}. \quad (38.4)$$

In Figs.3 and Fig.4 we display our numerical results for two different values of the grooves depth. We choose $h/2a = 0.1$ and $h/2a = 5$ to represent shallow and deep grooves respectively. The dots correspond to $h/2a = 0.1$ and the stars to $h/2a = 5$.

In Fig.3a we plot the ratio $\text{Re}\zeta_{22}^{(ef)}/\text{Re}\zeta$ versus a/b showing that the presence of the grooves leads to an increase of absorption. Our calculations show that if $h/2a < 1$, the value of $M(a/b)$ increases with an increase of the ratio h/a . However, the results for $M = M(a/b)$ are practically the same when $h/2a > 1$. This means that an incident wave "understands" the grooves, whose depth is more than $2a$, as infinitely deep. In other words, for our lamellar grating the stars in Fig.3a define the maximal values of the real part of the effective impedance in the s-polarization state.

In Fig.3b for the same values of $h/2a$ we present the values of $Z = \text{Re}\zeta_{22}^{(ef)}/\text{Re}\zeta_{11}^{(ef)}$ for different ratios a/b . The coefficient Z defines the distinction between the values of the effective impedance for the s- and p-polarization states. In Fig.3c we show the ratios Z_{pl}/Z_s versus a/b . This ratio defines the distribution of the absorbed energy between the horizontal sections $x_3 = 0$ and the inner surfaces of the grooves. In Fig.4 the coefficients $W = \text{Im}\zeta_{22}^{(ef)}/\text{Im}\zeta_{11}^{(ef)}$ versus a/b are plotted.

Our results show that the same as for p-polarized waves, the presence of the grating leads to an increase of absorption of incident s-polarized waves. However, since $Z < 1$, always the absorption of the p-polarized waves is more intensive. The more deep are the grooves, the more the difference between the polarizations. We also see (Fig.4) that the same as the factor Z , the factor W entering the expression for the imaginary part of the effective impedance (see Eq.(33.1)) is less than one. However, in contrast to Z , it depends on the ratio a/b only slightly.

We must note that a similar problem was solved in [17] by L.A.Vainshtein, S.M.Zhurav, A.I.Sukov. The authors of [17] examined a one-dimensional grating composed of semi-infinite plates (infinitely deep grooves) exposed to s-polarized normally incident electromagnetic wave. They also made use of the impedance boundary conditions (1) and with the aid of some other method based on the solution of the Wiener- Hopf equation, calculated the reflection coefficient R . Then supposing that $|\zeta|/kb \ll 1$, in the framework of the perturbation theory the value of R was defined up to the terms of the order of ζ . The real part of the effective impedance was determined when calculating the difference $1 - |R|^2$.

Of course, two different methods of calculation must lead to the same results. However, the values of $\text{Re}\zeta_{22}^{(ef)}/\text{Re}\zeta$ for deep ($h/2a > 1$) grooves obtained in the present work (the stars in Fig.3a) are approximately twenty percents less than the results of [17]. Trying to find out the origin of this discrepancy and to verify our approach, in [18] we examined the system investigated in [17] using two different approaches. We reproduced the perturbation theory calculations of [17] with the aid of the modal functions method used in the present work. The results were the same as in [17]. Then we repeated the calculation with the aid of Eq.(27.3). Comparing the results we obtained the same twenty percents difference. The analysis of [18] showed that the only source of the discrepancy could be in application of the perturbation theory.

Unfortunately, we were not successful in detecting accurately why the standard perturbation theory used in [17] and [18] led to the results differing from the results obtained with the aid of our approach based on Eq.(27.3). However, we found some reasons allowing us to assume that in this problem the standard perturbation theory was not justified properly (see [18]). We would like to note that the aforementioned difference is not of principle for qualitative description of the results obtained when analyzing the reflection of s-polarized waves from lamellar gratings.

5 SUMMARY

On the basis of the local impedance boundary conditions (1) we examined several metallic systems with strong surface inhomogeneities. For each system we calculated the effective surface impedance tensor.

Our results are exact within the accuracy of Eqs.(1).

If the surface of an inhomogeneous conductor is flat, the effective impedance is equal to the values of the local impedances averaged over the surface (see Eq.(7)). As an example, we examined a flat metallic surface with one-dimensional periodic strip-like local surface impedance. In this case the effective impedance is an isotropic tensor: $\zeta_{ik}^{(ef)} = \zeta_{ef} \delta_{ik}$ ($i, k = 1, 2$); ζ_{ef} is defined in Eq.(10).

We analyzed the role of evanescent waves paying special attention to the difference between the p- and s-polarization states. We showed that in the p-polarization state the evanescent waves did not contribute to the magnetic field in the immediate vicinity of the surface. On the contrary, in the s-polarization state just the evanescent waves provide the main contribution to the magnetic field taking account of the finite conductivity.

For this problem we also calculated the components of the time-averaged Poynting vector $\bar{\mathbf{S}}$. For normally incident waves the force lines of the vector $\bar{\mathbf{S}}$ being directed along the x_3 axis far from the surface ($x_3 > \lambda$) become distorted near the surface. The stronger is the surface inhomogeneity, the more the distortion of the lines. We showed that in the p-polarization state the Poynting vector was defined by the complex potential $w(z)$ ($z = x_1 + ix_3$) (see Eq.(19)). The aforementioned difference in the contribution of evanescent waves to the magnetic field is the reason why a complex potential cannot be introduced in the s-polarization state.

When the inhomogeneity is very strong (narrow strips with rather large impedance separated by wide strips with very small impedance), for both polarizations all the force lines of the vector $\bar{\mathbf{S}}$ meet at the surface in the narrow regions where the local impedance is big (see Fig.1). In the p-polarization state the equation for these lines is just the same as the equation for the force lines of the vector \mathbf{E} in the electrostatic problem for the periodic system of charged straight lines.

As an example of calculation of the effective impedance in the case of a conductor with rough surface we analyzed a periodic 1D surface of homogeneous isotropic conductor depicted in Fig.2a. In this case the effective impedance is a tensor whose elements differ significantly, when the grooves are rather deep.

In the p-polarization state the electromagnetic field penetrates into the grooves on the surface. In our frequency region this means that the magnetic field H_2^{per} is nearly constant in the immediate vicinity of the surface. As a result, the surface as a whole takes part in absorption of p-polarized waves. This leads to a significant enlargement of absorption: the longer the length of the surface profile, the more the real part of the element of the effective impedance tensor relating to the p-polarization state (see Eq.(31)). In the framework of our approximation the upper bound for the real part of this element of the effective impedance tensor is of the order of one. The surfaces showing the giant absorption of p-polarized waves ($\zeta_{ef} \sim 1$) can be realized, for example, if the contour of the grooves is a strongly branching line.

In the s-polarization state evanescent waves contribute to the magnetic field in the vicinity of the surface. As a result at the surface the components of the magnetic vector \mathbf{H}^{per} depend on coordinates. The most important that inside the grooves all the fields decay exponentially with the increasing distance from the plane $x_3 = 0$. No simple expressions for this element of the effective impedance tensor can be obtained. For each surface the calculation has to be done separately beginning from the calculation of magnetic field near the infinitely conducting surface of the same geometry.

As an example we examined the lamellar gratings shown in Fig.2c. Recently many authors studied scattering from surfaces having rectangular grooves. These surfaces are of interest because they can be manufactured very easily providing a possibility to check theoretical results. Most of the investigations have been devoted to resonant enhancement processes (see, for example, [12, 13, 16]). However, the frequency region examined in these works does not comply with the inequalities (2) defining the framework of the effective impedance approach applicability.

Our numerical results obtained when calculating the effective impedance associated with the lamellar grating for the s-polarization state are shown in Figs.3 and Fig.4.

Although in the s-polarization state the presence of the grating leads to an increase of the real part of the effective impedance too, even for very deep grooves it cannot achieve such giant values that are possible in the p-polarization state. Taking into account that in the case of an arbitrary 1-D surface profile s-polarized waves do not penetrate into deep grooves, we can generalize our result: if an arbitrary 1-D surface profile has rather deep grooves, the effective impedance associated with this surface is a strongly anisotropic tensor. Its element relating to the p-polarized state is much greater than the element relating to the s-polarized state.

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6 APPENDIX 1

Herein for the p-polarization state we calculate the leading term in the expression for the total magnetic field H_2^{per} above the infinitely conducting lamellar grating depicted in Fig.2c supposing that $h, b \ll \lambda$.

In the half-space above the grooves ($x_3 > 0$) we adopt the plane-wave representation of the magnetic field $H_2^{per} = H_2^+(x_1, x_3)$:

$$H_2^+(x_1, x_3) = H_0 \left\{ e^{-ikx_3} + \sum_{q=-\infty}^{\infty} h_q^+ e^{i(\pi qx_1/b + \alpha_q x_3)} \right\}, \quad (A.1)$$

α_q is defined in Eq.(34.2); H_0 is the amplitude of the incident wave.

Inside the central groove ($|x_1| < a$; $-h < x_3 < 0$) we seek $H_2^{per} = H_2^-(x_1, x_3)$ as a series of modal functions $\phi_n^{(p)}(x_1, x_3)$ that are the solutions of the Maxwell equations. With regard to the boundary conditions for perfect conductors we have

$$H_2^-(x_1, x_3) = H_0 \sum_{n=0}^{\infty} B_n^{(p)} \phi_n^{(p)}(x_1, x_3); \quad \phi_n^{(p)}(x_1, x_3) = \cos\left(\frac{\pi n}{2a}(x_1 - a)\right) \cos \beta_n(h + x_3), \quad (A.2)$$

β_n is defined in Eq.(34.1).

Applying the boundary conditions on the plateaus $x_3 = 0$ and the continuity conditions across the central slit, we obtain the set of equations for the coefficients $B_n^{(p)}$:

$$i \sum_{n=0}^{\infty} (\beta_n b) B_n^{(p)} \sin(\beta_n h) I_{nm}^{(p)} - \frac{a}{2b} [B_m^{(p)} \cos \beta_m h + \delta_{m0} B_0^{(p)} \cos(kh)] = -2C_{m0}^*, \quad (A.3a)$$

$$I_{nm}^{(p)} = \sum_{q=-\infty}^{\infty} \frac{1}{(\alpha_q b)} C_{nq} C_{mq}^*, \quad (A.3b)$$

$$C_{nq} = 2i \frac{a}{b} A(m; 2qa/b), \quad A(m; z) = \frac{z}{2\pi} \frac{[e^{-i\pi z/2} - (-1)^m e^{i\pi z/2}]}{[z^2 - m^2]}, \quad (A.3c)$$

In terms of the coefficients $B_n^{(p)}$ the amplitudes h_q^+ are:

$$h_q^+ = \delta_{q0} + \frac{i}{\alpha_q} \sum_{n=0}^{\infty} B_n^{(p)} \beta_n C_{nq} \sin \beta_n h. \quad (A.4)$$

When the elements of the matrix $I_{nm}^{(p)}$ are presented as the series expansions in powers of the small parameter $\gamma = kb$, it is easy to see that the element $I_{00}^{(p)}$ has the term $a/2b\gamma$, and the expansions of all the other elements begin with the terms independent of γ . Then from Eqs.(A.3a) it follows that up to the terms independent of γ

$$B_0^{(p)} = 2; \quad B_n^{(p)} = 0, \text{ if } n \neq 0. \quad (A.5a)$$

Within the same accuracy from Eq.(A.5) we have

$$h_0^+ = 1; \quad h_q^+ = 0, \text{ if } q \neq 0. \quad (A.5b)$$

Thus, inside the grooves the modal function with $n = 0$ gives the main contribution to the magnetic field. Next, the amplitudes of evanescent waves are negligibly small in comparison with the amplitude of the reflected wave. With regard to Eq.(A.1) and Eq.(A.2) this means that in the vicinity of the surface the magnetic field is nearly constant: $H_2^{per}(x_1, x_3) \approx 2H_0$.

7 APPENDIX 2

In the s-polarization state when calculating the components of the magnetic vector \mathbf{H}^{per} , special attention has to be drawn to the corner points of our lamellar grating. It is well known that if a surface has geometrical singularities, some components of electromagnetic vectors have singularities too. In the s-polarization state the tangential magnetic field in the immediate vicinity of a rectangular two-dimensional infinitely conducting wedge (the wedge is along the x_2 direction) is proportional to $\rho^{-1/3}$, where $\rho = \sqrt{x_1^2 + x_3^2}$ is the distance from the corner point (see, for example, [15]). Consequently, $H_1^{(+)}(x_1, 0) \sim |x_1 \pm a|^{-1/3}$, when $x_1 \rightarrow \pm a$, and $H_3^{(-)}(\pm a, x_3)$, when $x_3 \rightarrow 0$. We need the solution of Eqs.(36) providing the aforementioned behavior of the magnetic field.

In [15] it was shown that the behavior of the fields near the edge is defined by the asymptotic behavior of the modal functions representing the fields. Let us show that in our case the tangential magnetic field near the edge is proportional to $\rho^{-1/3}$, if $Y_n \sim 1/(2n-1)^{2/3}$, when $n \rightarrow \infty$.

Indeed, let us examine, for example, $H_3^{(-)}(x_1, x_3)$ when $x_1 = a$ and $x_3 \rightarrow -0$. It is the tangential magnetic field on the vertical facet of the groove near its throat. Taking into account that when $x_3 \rightarrow 0$, the behavior of $H_3^{(-)}(a, x_3)$ is defined by the coefficients Y_n with $n \rightarrow \infty$, with regard to Eq.(37.4) we obtain

$$H_3^{(-)}(a, x_3) \approx \frac{\pi b}{2a} \sum_{n=1}^{\infty} Y_n e^{-n|x_3|}, \quad |x_3| \rightarrow 0. \quad (\text{A.6a})$$

Suppose $Y_n \sim (2n-1)^p$ when $n \rightarrow \infty$. With regard to the asymptotic equality [15]

$$\sum_{n=1}^{\infty} n^p e^{-nz} \approx \Gamma(1+p) z^{-(p+1)}, \quad z \rightarrow +0 \text{ and } -1 < p < 0, \quad (\text{A.6b})$$

($\Gamma(x)$ is the Gamma function), comparing equations (A.6a) and (A.6b) we find that $p = -2/3$ has to be chosen.

The next step is to search the solution of the set (36.1) truncating it to finite order N and increasing N until stability of successive solutions is obtained. Simultaneously, the sum defining the matrix Δ_{nm} (see Eq.(36.2) has to be truncated to the finite order Q . The solution has to guarantee that $Y_n \sim (2n-1)^{-2/3}$, when $n \gg 1$.

In [15] a similar set of equations was analyzed. This set was obtained when calculating the fields inside a rectangular one-dimensional branching wave-guide. It was shown that such set of equations allowed unrestricted number of solutions depending on the limit of the ratio Q/N for $Q, N \rightarrow \infty$. The value of this limit ensuring the necessary behavior of the fields near the edge of the wedge determined the physically meaningful solution of the set in question. For a branching wave-guide this limit was calculated analytically as a function of the ratio of the sizes characterizing the wave-guide.

Unfortunately, in the case of the lamellar grating we failed trying to define analytically the limit $\lim_{N, Q \rightarrow \infty} (Q/N)$ providing $Y_n \sim (2n-1)^{-2/3}$ for $n \rightarrow \infty$. Our results are based on numerical calculations.

In all the calculations we used $N = 40$. For this truncation number the stabilization was obtained in all examined examples. This value of N was sufficient to obtain a stable exponential solution $Y_n \sim ((2n-1)^{-\tau})$ for $n \gg 1$ as well as the stable results for the real and imaginary parts of the effective impedance calculated with the aid of (33.1) and (38).

To determine the ratio Q/N for given values of h/a and a/b , we solved the set of equations (36) using different truncation numbers Q and chose Q ensuring $\tau = 2/3$. We find that the ratio Q/N providing the given value of τ , depends on the ratio h/a only slightly. When calculating Q/N as function of a/b we saw that rather accurately the value $\tau = 2/3$ corresponded to the constant value of the ratio aQ/bN . For $h/a = .2$ and $h/a = 2$ the values of aQ/bN were 1.9 and 2.1 respectively. In Fig.5 for $a/b = 1/\sqrt{2}$ and $h = 2a$ we show the exponent τ as function of x : $Q = [xbN/a]$ ($[z]$ denotes the integral part of the number z).

It worth to be mentioned that although for $n \gg 1$ the behavior of the coefficients Y_n depends on the $\lim_{N, Q \rightarrow \infty} (Q/N)$, for small numbers n the coefficients Y_n are practically the same when $Qa/Nb > 1$.

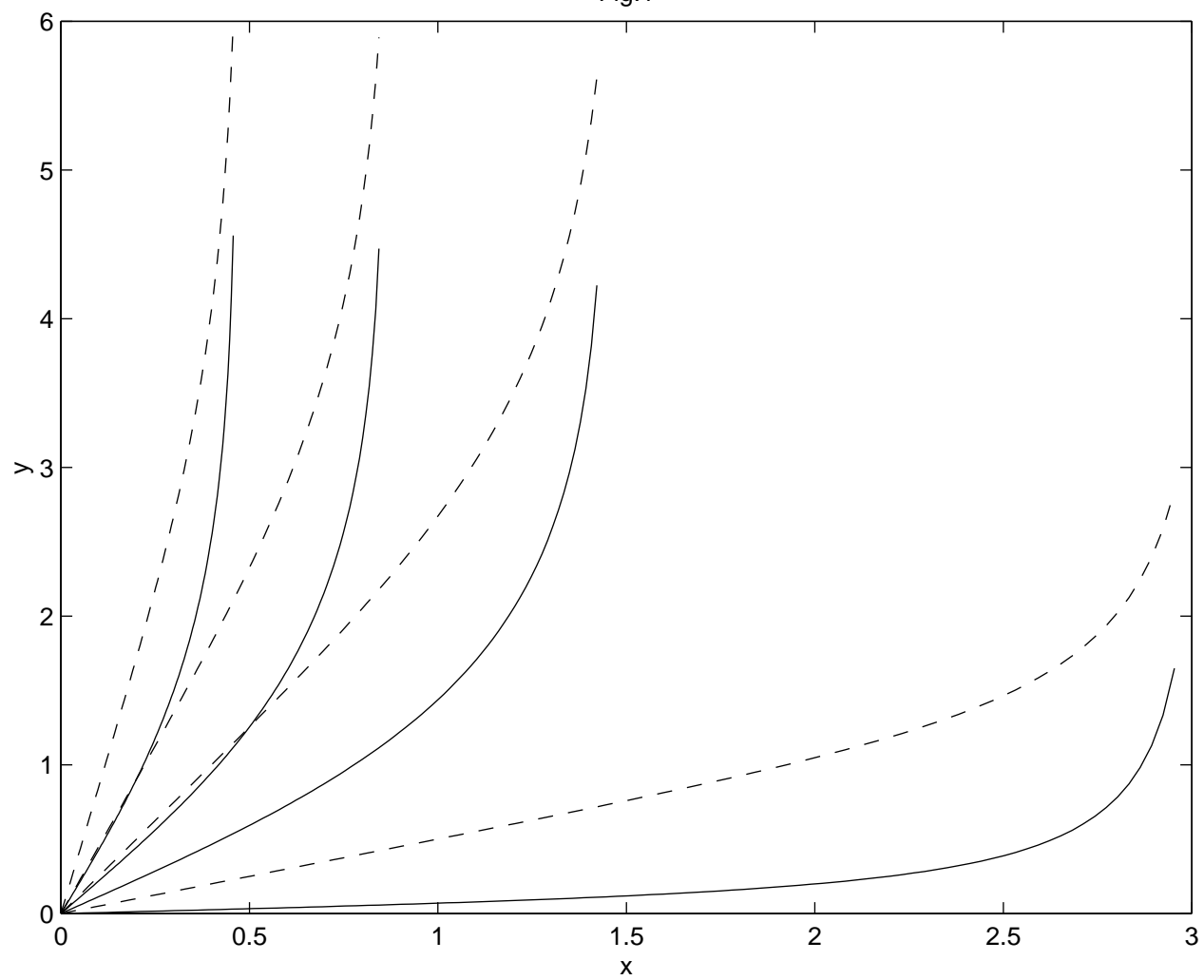
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Fig.1



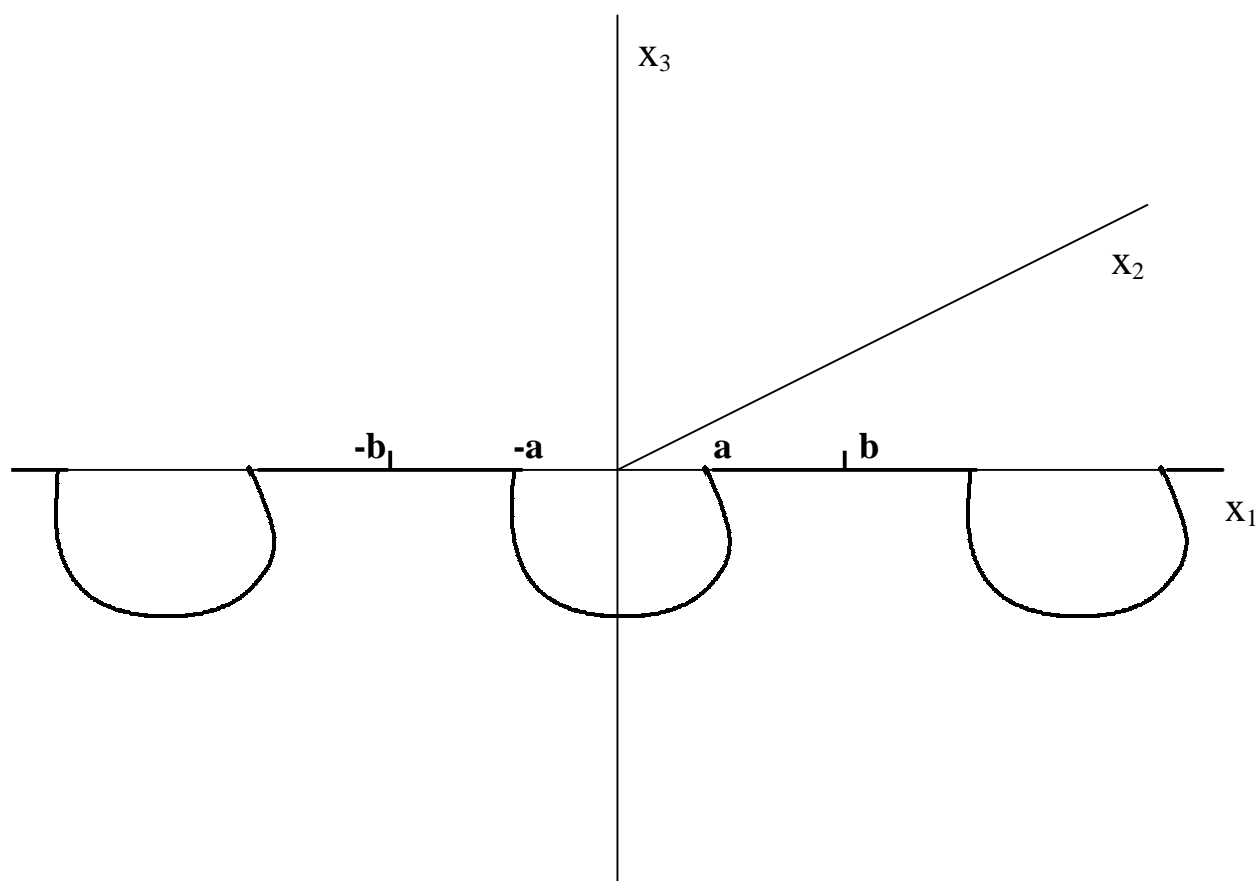


Fig. 2 a

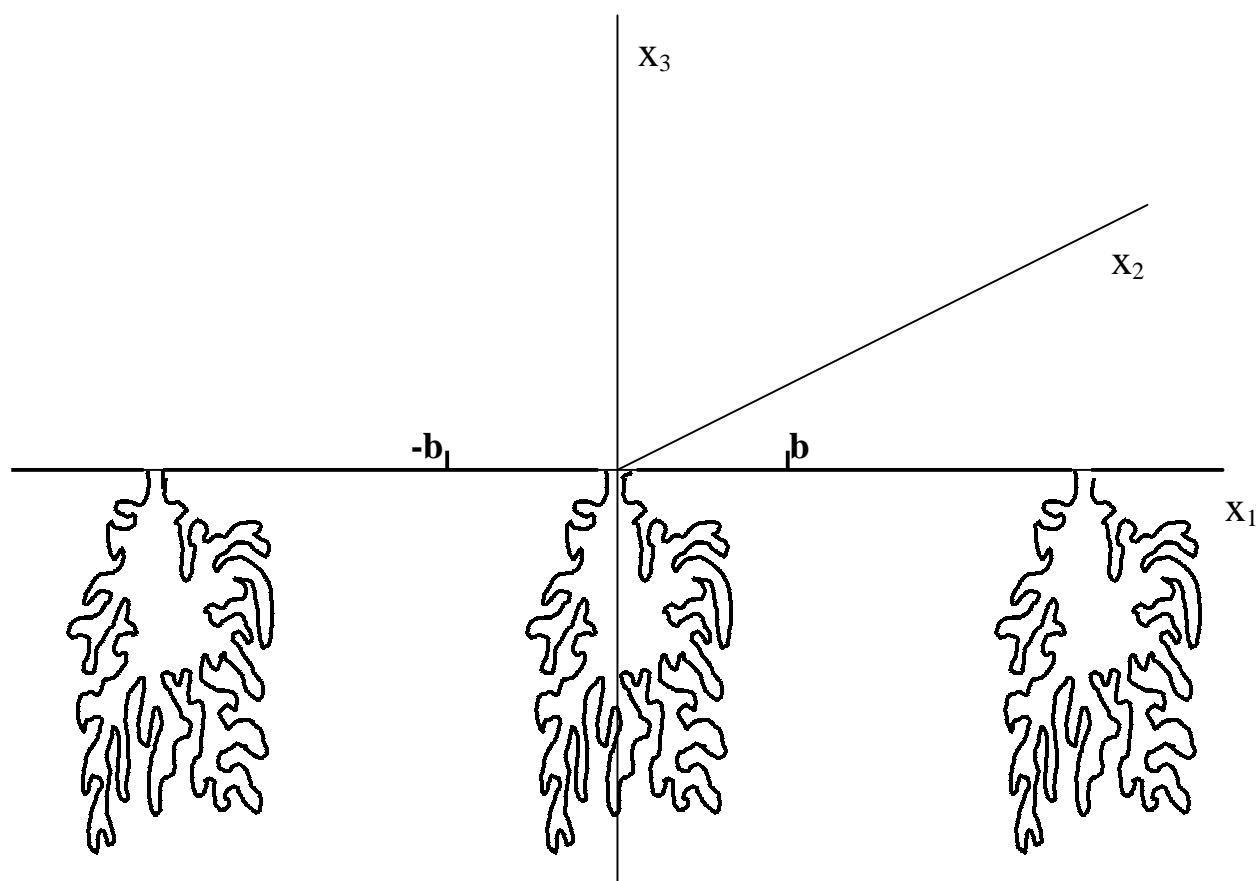


Fig. 2 b

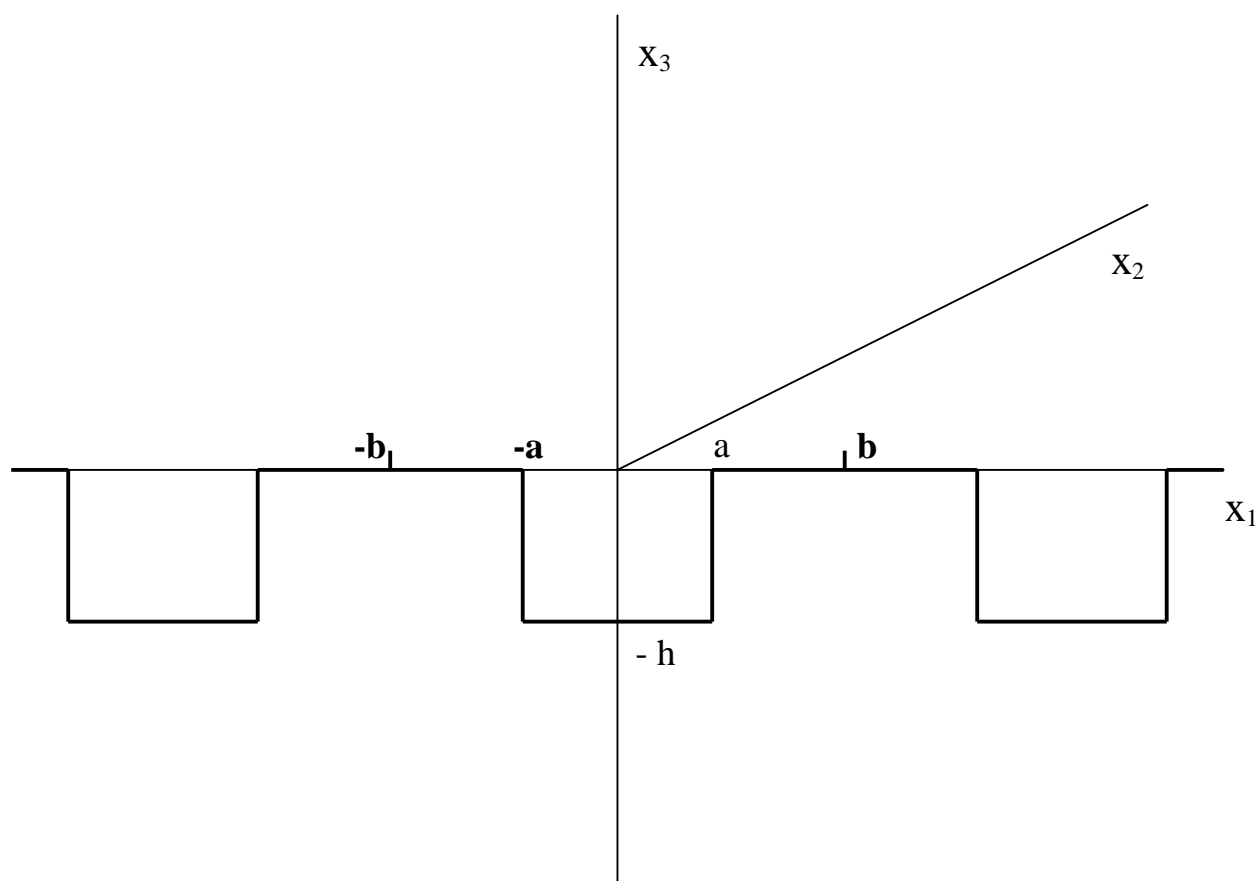


Fig. 2 c

Fig.3a

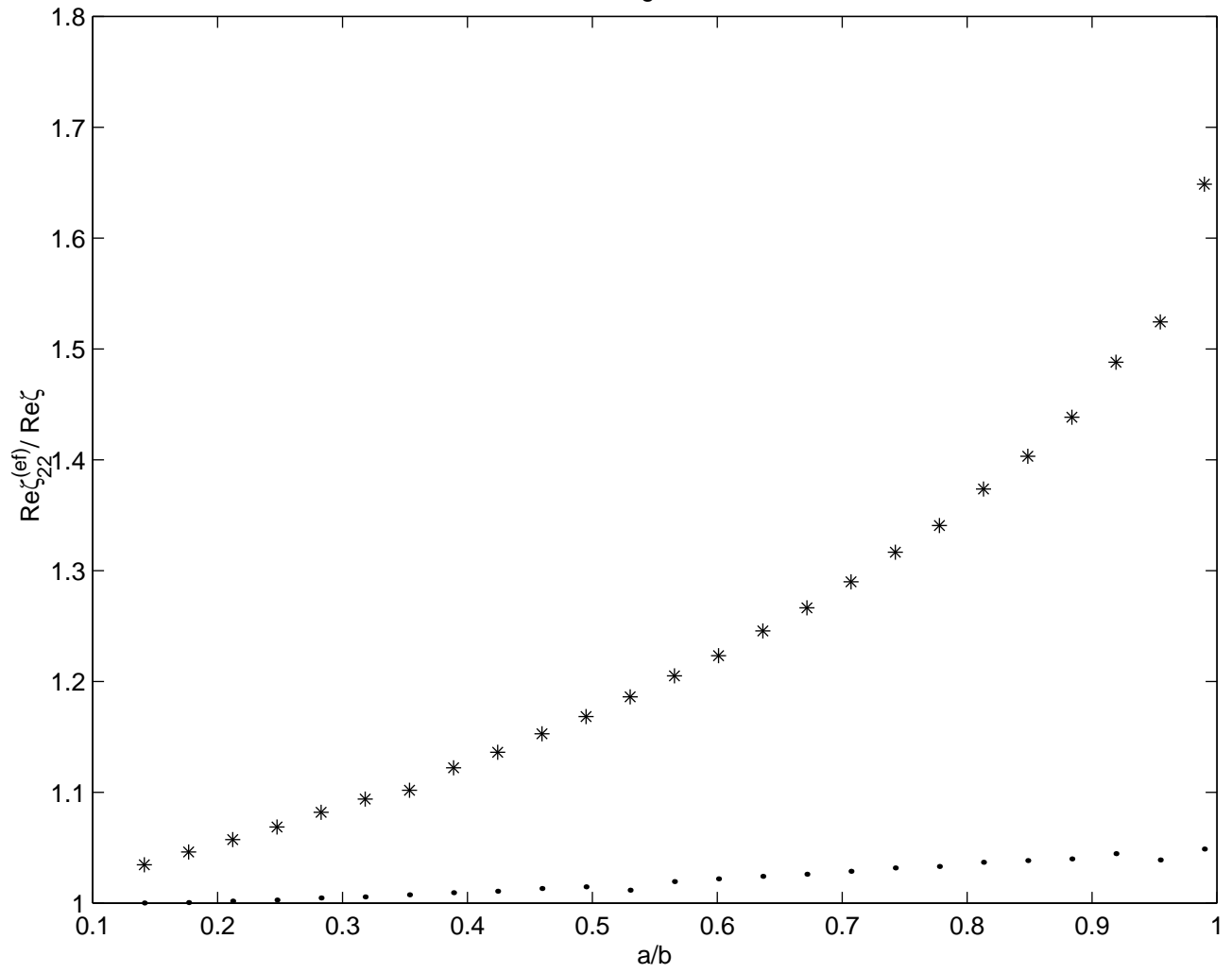


Fig.3b

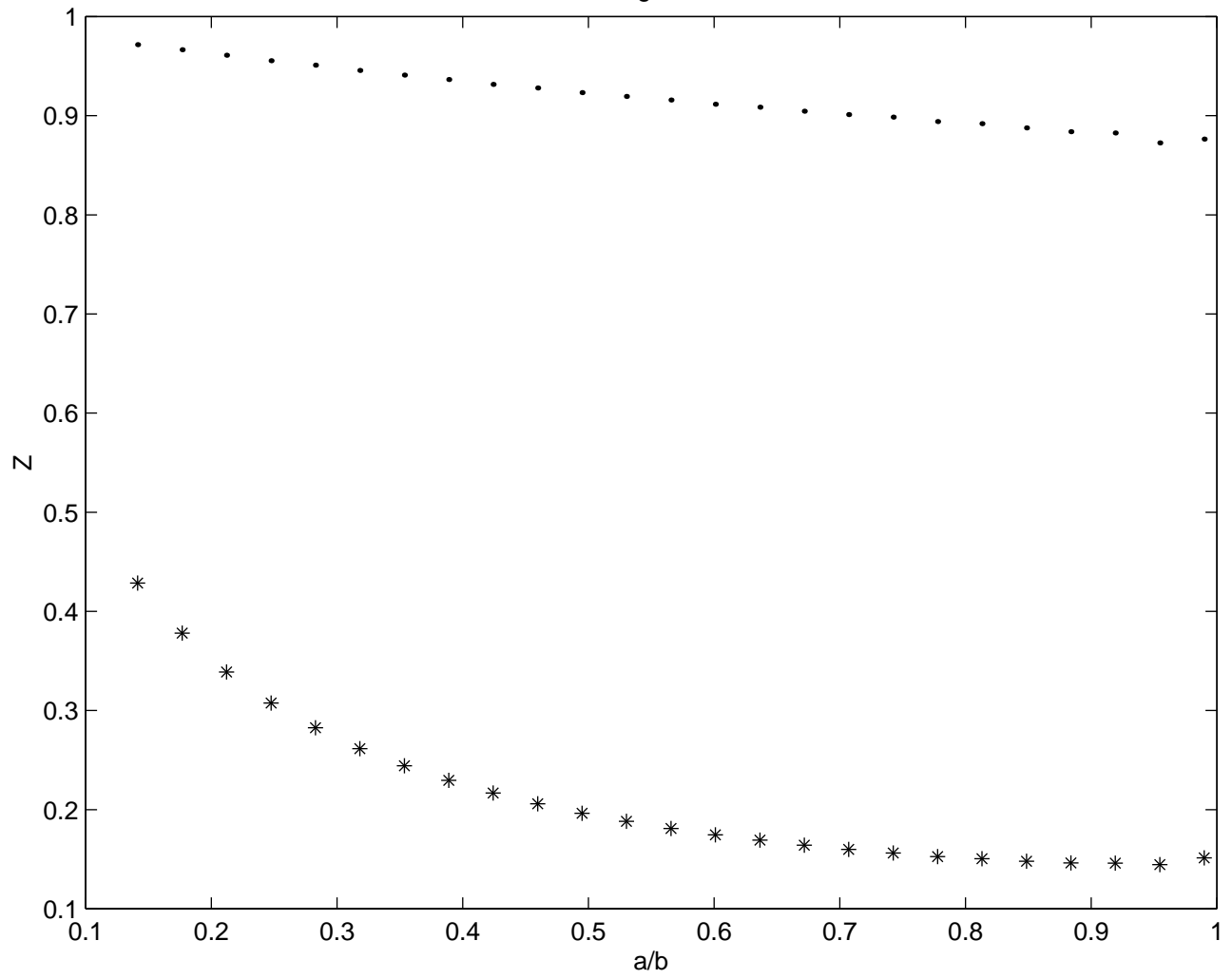


Fig.3c

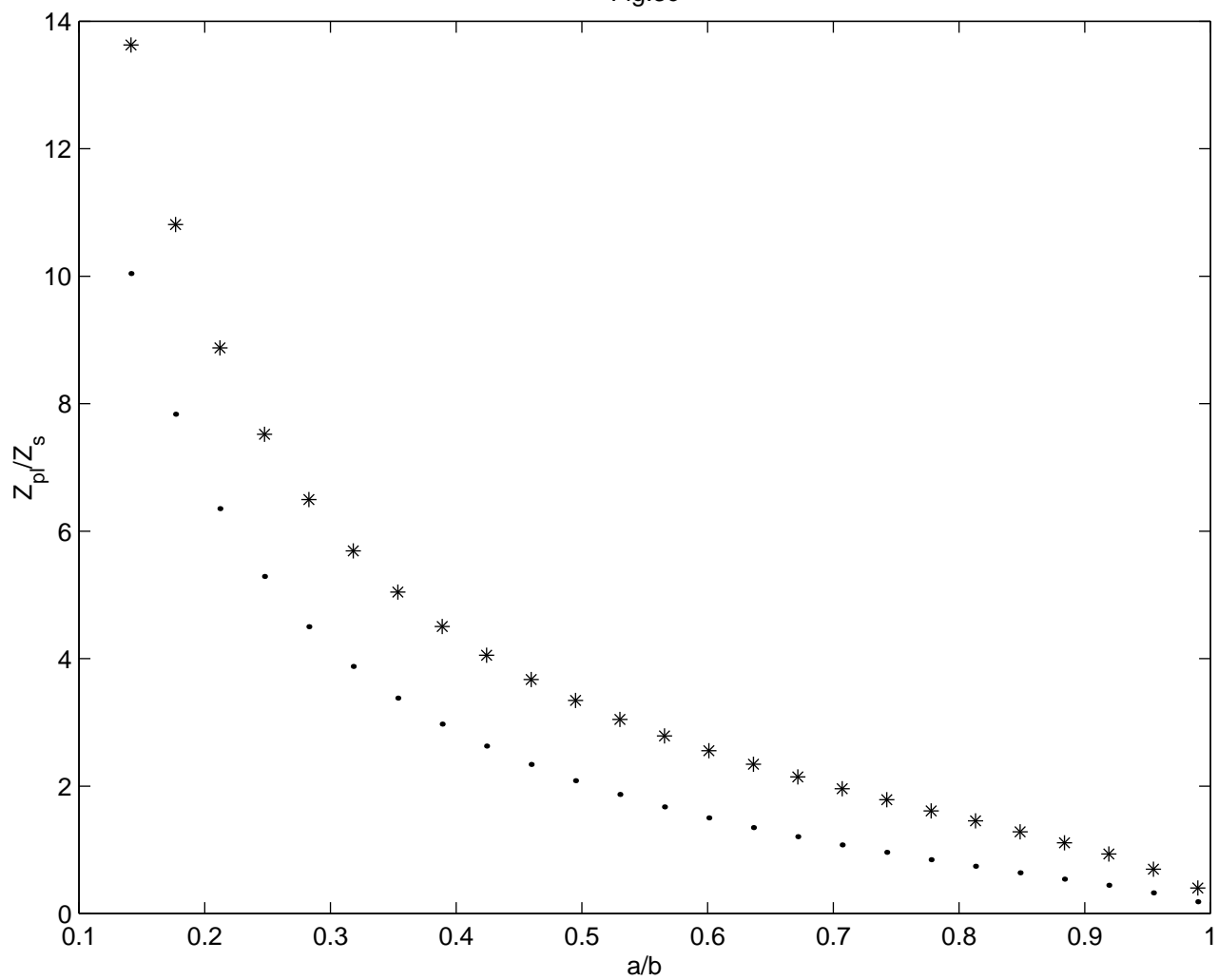


Fig.4

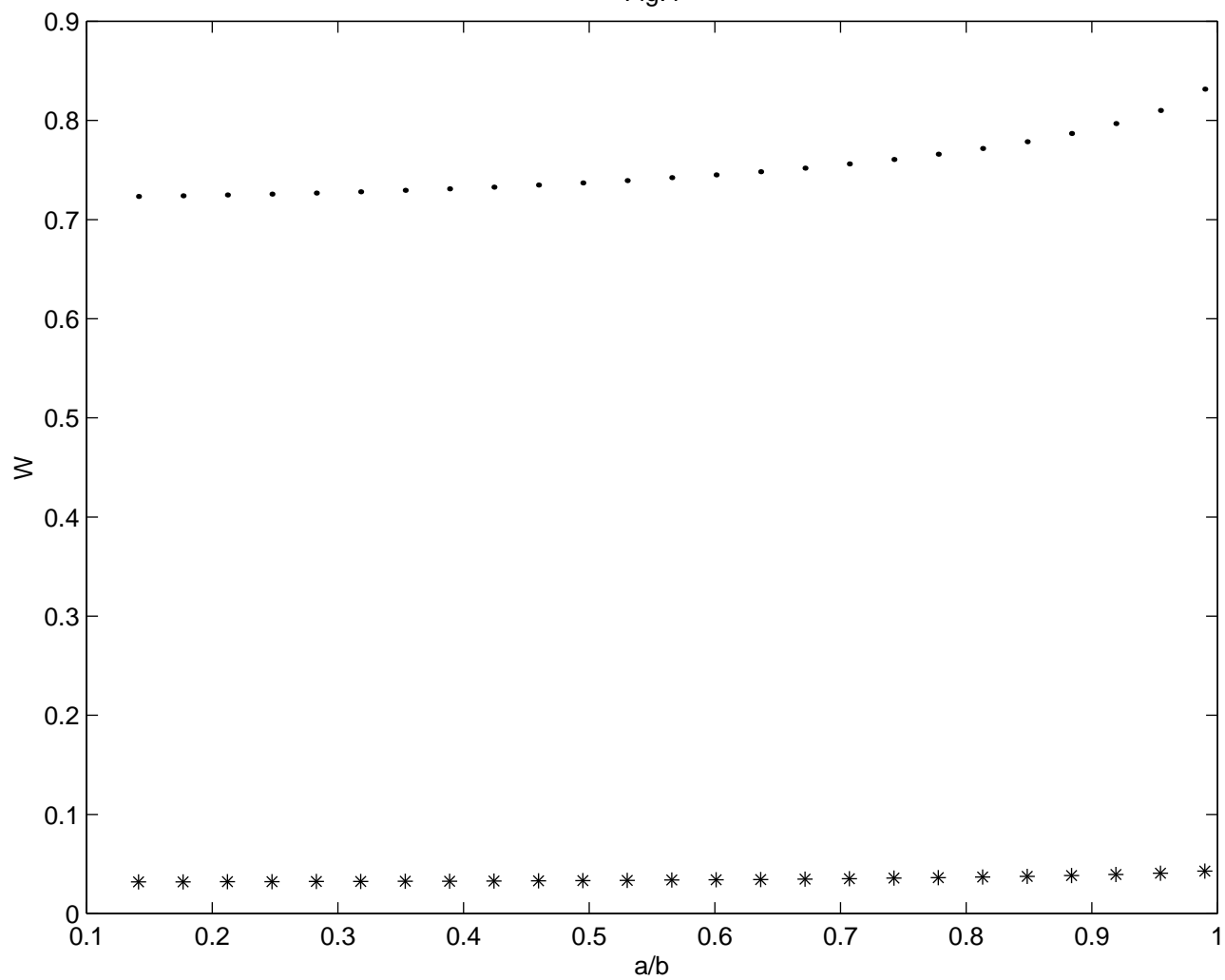


Fig.5

